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Algebraic Bethe ansatz for the one-dimensional Hubbard model with open boundaries

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Abstract. The one-dimensional Hubbard model with open boundary conditions is exactly solved by means of the algebraic Bethe ansatz. The eigenvalue of the transfer matrix and the energy spectrum, as well as the Bethe ansatz equations, are obtained.

1. Introduction

The one-dimensional (1D) Hubbard model has been one of the most fundamental and favoured integrable models in non-perturbative quantum field theory. It exhibits on-site Coulomb interaction and correlated hopping which might possibly reveal a promising role in understanding the mystery of high-T_c superconductivity. Since Lieb and Wu [1], in 1968, solved the 1D Hubbard model with periodic boundary conditions (BC) using the coordinate Bethe ansatz, there has been a great deal of work devoted to the study of this model. A remarkable step was performed by Shastry [2] who showed that the Hamiltonian of the 1D Hubbard periodic chain commutes with a one-parameter family of transfer matrices of an equivalent coupled symmetric XY spin chain and who also gave a direct proof of the integrability of the model by presenting the quantum R-matrix. Later on, Wadati and coworkers [3, 4] further studied its integrability in terms of the quantum inverse scattering method (QISM) [5, 6]. Very recently, Martins and Ramos [7] proposed a useful way of solving the eigenvalue problem of the transfer matrix of the 1D Hubbard model with periodic BC by means of the algebraic Beth ansatz. Their approach provides us with a unified way to solve a wide class of Hubbard-like models [8, 9] by the algebraic Bethe ansatz.

On the other hand, in recent years, there has been much interest in the study of the quantum integrable systems with open BC, i.e., the systems of finite interval with independent BC on each end. The presence of the boundary fields which lead to a pure back scattering on each end of the quantum chain and the exhibition of the quantum group symmetry by special choice of the boundary parameters endow the system with rich physical properties [10–12] from a thermodynamical point of view. A systematic approach to handle the open BC for 1D integrable quantum chains was proposed by Sklyanin [13]. A further extension of Sklyanin's formalism to deal with a more general class of models associated with Lie (super) algebras was proposed by Mezincescu and Nepomechie [14]. The graded version of the boundary QISM was worked out in [15,16]. We also remark that the coordinate Bethe ansatz for the 1D Hubbard model with integrable BC was studied in [19]. At present, though there are several authors [15, 17–19] who have studied the open BC for the 1D Hubbard model, the algebraic Bethe ansatz solution has not yet been achieved. Actually, the diagonalization of the transfer matrix which provide us with the spectrum of all conserved charges should be more essential in studying the finite-temperature properties of the integrable models [20,21] than diagonalization of the underlying Hamiltonian. But, as we know, the reflection equations for the 1D Hubbard model are much more involved and the quantum R-matrix does not have the additive property that makes it difficult to build up the necessary commutation rules among the diagonal and creation fields. In this paper, we intend to generalize Sklyanin's formalism to solve the 1D Hubbard model with open BC. The eigenvalue of the transfer matrix and Bethe ansatz equations for the model is given. It is found that the model exhibits a hidden *XXX* spin open chain which plays a crucial role in solving the model.

This paper is organized as follows. In section 2, we shall recall the main results about open BC for the 1D Hubbard model in order to introduce the notation which is used in this paper. In section 3, we perform the algebraic Bethe ansatz approach for the model. In section 4, we formulate the nested algebraic Bethe ansatz for the hidden *XXX* quantum spin open chain and present our main results. Section 5 is devoted to the conclusions.

2. The 1D Hubbard model with boundary fields

Let us consider the 1D Hubbard model with boundary fields determined by the Hamiltonian [15, 17, 18]

$$H = -\sum_{j=1}^{N-1}\sum_{s}(a^\dagger_{j+1s}a_{js} + a^\dagger_{js}a_{j+1s}) + U\sum_{j=1}^{N}(n_{j\uparrow} - \tfrac{1}{2})(n_{j\downarrow} - \tfrac{1}{2}) \\ + p_+(2n_{1\uparrow} - 1) + p_-(2n_{1\downarrow} - 1) + q_+(2n_{N\uparrow} - 1) + q_-(2n_{N\downarrow} - 1). \tag{2.1}$$

Here $p_\pm$ and $q_\pm$ are the free boundary parameters characterizing the boundary fields. The coupling U describes the on-site Coulomb interaction and $a^\dagger_{js}$ and a_{js} are creation and annihilation operators with spins ($s = \uparrow$ or $\downarrow$) at site j satisfying the anti-commutation relations

$$\{a_{js}, a_{j's'}\} = \{a^\dagger_{js}, a^\dagger_{j's'}\} = 0 \tag{2.2}$$

$$\{a_{js}, a^\dagger_{j's'}\} = \delta_{jj'}\delta_{ss'} \tag{2.3}$$

and $n_{js} = a^\dagger_{js}a_{js}$ is the density operator. The Lax operator is given [3,4] by

$$\mathcal{L}_j(u) = \begin{pmatrix} -e^{h(u)}f_{j\uparrow}f_{j\downarrow} & -f_{j\uparrow}a_{j\downarrow} & \mathrm{i}a_{j\uparrow}f_{j\downarrow} & \mathrm{i}e^{h(u)}a_{j\uparrow}a_{j\downarrow} \\ -\mathrm{i}f_{j\uparrow}a^\dagger_{j\downarrow} & e^{-h(u)}f_{j\uparrow}g_{j\downarrow} & e^{-h(u)}a_{j\uparrow}a^\dagger_{j\downarrow} & \mathrm{i}a_{j\uparrow}g_{j\downarrow} \\ a^\dagger_{j\uparrow}f_{j\downarrow} & e^{-h(u)}a^\dagger_{j\uparrow}a_{j\downarrow} & e^{-h(u)}g_{j\uparrow}f_{j\downarrow} & g_{j\uparrow}a_{j\downarrow} \\ -\mathrm{i}e^{h(u)}a^\dagger_{j\uparrow}a^\dagger_{j\downarrow} & a^\dagger_{j\uparrow}g_{j\downarrow} & \mathrm{i}g_{j\uparrow}a^\dagger_{j\downarrow} & -e^{h(u)}g_{j\uparrow}a_{j\downarrow} \end{pmatrix} \tag{2.4}$$

where

$$f_{js} = \sin u - (\sin u - \mathrm{i}\cos u)n_{js} \qquad g_{js} = \cos u - (\cos u + \mathrm{i}\sin u)n_{js}.$$

With the grading $P(1) = P(4) = 0$, $P(2) = P(3) = 1$ and the constraint condition

$$\frac{\sinh 2h(u)}{\sin 2u} = \frac{U}{4} \tag{2.5}$$

the Lax operator (2.4) satisfies the graded Yang–Baxter algebra

$$\mathcal{R}_{12}(u,v)\overset{1}{\mathcal{T}}(u)\overset{2}{\mathcal{T}}(v) = \overset{2}{\mathcal{T}}(v)\overset{1}{\mathcal{T}}(u)\mathcal{R}_{12}(u,v) \tag{2.6}$$

where the monodromy matrix $\mathcal{T}(u)$ is defined by

$$\mathcal{T}(u) = \mathcal{L}_N(u)\ldots\mathcal{L}_1(u) \tag{2.7}$$

and

$$\overset{1}{\mathcal{T}}(u) = \mathcal{T}(u)\otimes_s I \qquad \overset{2}{\mathcal{T}}(u) = I\otimes_s \mathcal{T}(u). \tag{2.8}$$

here $\otimes_s$ is the super direct product

$$[A\otimes_s B]_{\alpha\beta,\gamma\delta} = (-1)^{[P(\alpha)+P(\gamma)]P(\beta)}A_{\alpha\gamma}B_{\beta\delta}.$$

For our convenience in practical calculation, we display the associated quantum $\mathcal{R}_{12}(u,v)$-matrix in the appendix. One may show that $\mathcal{R}_{12}(u,v)$ enjoys the following graded reflection equations (RE) [18]:

$$\mathcal{R}_{12}(u,v)\overset{1}{K}_{-}(u)\mathcal{R}_{21}(v,-u)\overset{2}{K}_{-}(v) = \overset{2}{K}_{-}(v)\mathcal{R}_{12}(u,-v)\overset{1}{K}_{-}(u)\mathcal{R}_{21}(-v,-u) \tag{2.9}$$

$$\begin{aligned}&\mathcal{R}_{21}^{\mathrm{St}_1\overline{\mathrm{St}}_2}(v,u)\overset{1}{K}{}_{+}^{\mathrm{St}_1}(u)\mathcal{R}_{12}^{\mathrm{St}_1\overline{\mathrm{St}}_2}(-u,v)\overset{2}{K}{}_{+}^{\overline{\mathrm{St}}_2}(v)\\ &\quad= \overset{2}{K}{}_{+}^{\overline{\mathrm{St}}_2}(v)\mathcal{R}_{21}^{\mathrm{St}_1\overline{\mathrm{St}}_2}(-v,u)\overset{1}{K}{}_{+}^{\mathrm{St}_1}(u)\mathcal{R}_{12}^{\mathrm{St}_1\overline{\mathrm{St}}_2}(-u,-v)\end{aligned} \tag{2.10}$$

which ensure the integrability of the model (2.1), provided that

$$K_{\pm}(u) = \begin{pmatrix} K1_{\pm}(u) & 0 & 0 & 0\\ 0 & K2_{\pm}(u) & 0 & 0\\ 0 & 0 & K3_{\pm}(u) & 0\\ 0 & 0 & 0 & K4_{\pm}(u)\end{pmatrix} \tag{2.11}$$

where $p_+ = p_- = \xi_-/2$

$$\begin{aligned}
K1_-(u) &= \lambda_-(\mathrm{e}^{-h(u)}\cos u - \mathrm{e}^{h(u)}\xi_-\sin u)(\mathrm{e}^{h(u)}\cos u - \mathrm{e}^{-h(u)}\xi_-\sin u)\\
K2_-(u) &= \lambda_-(\mathrm{e}^{-h(u)}\cos u + \mathrm{e}^{h(u)}\xi_-\sin u)(\mathrm{e}^{-h(u)}\cos u - \mathrm{e}^{h(u)}\xi_-\sin u)\\
K3_-(u) &= \lambda_-(\mathrm{e}^{-h(u)}\cos u + \mathrm{e}^{h(u)}\xi_-\sin u)(\mathrm{e}^{-h(u)}\cos u - \mathrm{e}^{h(u)}\xi_-\sin u)\\
K4_-(u) &= \lambda_-(\mathrm{e}^{h(u)}\cos u + \mathrm{e}^{-h(u)}\xi_-\sin u)(\mathrm{e}^{-h(u)}\cos u + \mathrm{e}^{h(u)}\xi_-\sin u)
\end{aligned} \tag{2.12}$$

and $q_+ = q_- = \xi_+/2$

$$\begin{aligned}
K1_+(u) &= \lambda_+(\mathrm{e}^{-h(u)}\xi_+\cos u + \mathrm{e}^{h(u)}\sin u)(\mathrm{e}^{h(u)}\xi_+\cos u + \mathrm{e}^{-h(u)}\sin u)\\
K2_+(u) &= \lambda_+(\mathrm{e}^{h(u)}\xi_+\cos u + \mathrm{e}^{-h(u)}\sin u)(\mathrm{e}^{h(u)}\xi_+\cos u - \mathrm{e}^{-h(u)}\sin u)\\
K3_+(u) &= \lambda_+(\mathrm{e}^{h(u)}\xi_+\cos u + \mathrm{e}^{-h(u)}\sin u)(\mathrm{e}^{h(u)}\xi_+\cos u - \mathrm{e}^{-h(u)}\sin u)\\
K4_+(u) &= \lambda_+(\mathrm{e}^{h(u)}\xi_+\cos u - \mathrm{e}^{-h(u)}\sin u)(\mathrm{e}^{-h(u)}\xi_+\cos u - \mathrm{e}^{h(u)}\sin u).
\end{aligned} \tag{2.13}$$

Here $\lambda_\pm$ and $\xi_\pm$ are arbitrary constants describing boundary effects. $\overline{\mathrm{St}}_a$ stands for the inverse of the supertransposition in the space a. The supertransposition is defined by

$$(A_{ij})^{\mathrm{St}} = (-1)^{[P(i)+1]P(j)}A_{ji}.$$

We would like to remark that Zhou [15] first gave the boundary $K_\pm$-matrices equivalent to (2.12) and (2.13) in terms of QISM. Consequently, using Lax pair formulation, the authors of [17] presented two classes of boundary $K_\pm$-matrices, leading to four possible boundary terms in the 1D Hubbard open chain Hamiltonian, while Shiroishi and Wadati [18] studied the open BC for the model in terms of the graded version of QISM and also presented two classes of the solutions to the graded RE. The second solution to the graded RE (2.9) and (2.10) permits the boundary fields with $p_+ = -p_-$ and $q_+ = -q_-$ corresponding to magnetic boundary fields (see [15, 17, 18]). In this paper, we restrict ourselves to studying the chemical boundary fields (2.12) and (2.13) based on the consideration that this kind of BC will bring us a simpler boundary K-matrix for the hidden XXX open chain. For other kinds of BC, of course, we may treat them in a similar way. It is found that the Hamiltonian (2.1) is related to the double-row monodromy matrix

$$\tau(u) = \mathrm{Str}_0 K_+(u)\mathcal{T}(u)K_-(u)\mathcal{T}^{-1}(-u) \tag{2.14}$$

in the following way:

$$\tau(u) = c_1 u + c_2 u^2 + c_3(H + \mathrm{const})u^3 + \cdots \tag{2.15}$$

where c_i, $i = 1, \ldots, 4$, are some scalar functions of boundary parameters. Str_0 denotes the supertrace carried out in auxiliary space v_0.

3. Algebraic Bethe ansatz approach

According to the algebraic Bethe ansatz, let us first choose the standard ferromagnetic pseudovacuum state $|0\rangle_i$

$$|0\rangle_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \otimes_s \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \tag{3.1}$$

as a highest vector, which corresponds to the doubly occupied state. Following the notation introduced in [22], we define the monodromy matrix $\mathcal{T}(u)$ as

$$\mathcal{T}(u) = \begin{pmatrix} B(u) & B_1(u) & B_2(u) & F(u) \\ C_1(u) & A_{11}(u) & A_{12}(u) & E_1(u) \\ C_2(u) & A_{21}(u) & A_{22}(u) & E_2(u) \\ C_3(u) & C_4(u) & C_5(u) & D(u) \end{pmatrix} \tag{3.2}$$

$$\mathcal{T}^{-1}(-u) = \begin{pmatrix} \overline{B}(u) & \overline{B}_1(u) & \overline{B}_2(u) & \overline{F}(u) \\ \overline{C}_1(u) & \overline{A}_{11}(u) & \overline{A}_{12}(u) & \overline{E}_1(u) \\ \overline{C}_2(u) & \overline{A}_{21}(u) & \overline{A}_{22}(u) & \overline{E}_2(u) \\ \overline{C}_3(u) & \overline{C}_4(u) & \overline{C}_5(u) & \overline{D}(u) \end{pmatrix} \tag{3.3}$$

and

$$\mathcal{T}_-(u) = \mathcal{T}(u)K_-(u)\mathcal{T}^{-1}(-u)$$
$$= \begin{pmatrix} \tilde{B}(u) & \tilde{B}_1(u) & \tilde{B}_2(u) & \tilde{F}(u) \\ \tilde{C}_1(u) & \tilde{A}_{11}(u) & \tilde{A}_{12}(u) & \tilde{E}_1(u) \\ \tilde{C}_2(u) & \tilde{A}_{21}(u) & \tilde{A}_{22}(u) & \tilde{E}_2(u) \\ \tilde{C}_3(u) & \tilde{C}_4(u) & \tilde{C}_5(u) & \tilde{D}(u) \end{pmatrix}. \tag{3.4}$$

It is not difficult to show that $\mathcal{T}_-(u)$ also satisfies the RE (2.9). With $\mathcal{T}(u)$ and $\mathcal{T}^{-1}(-u)$ acting on the pseudovacuum state

$$|0\rangle = \otimes_{i=1}^N |0\rangle_i \tag{3.5}$$

we have the following properties (upon a common factor):

$$
\begin{aligned}
&B(u)|0\rangle = \overline{B}(u)|0\rangle = \left\{\frac{\cos u}{\sin u}\mathrm{e}^{2h(u)}\right\}^N |0\rangle \\
&D(u)|0\rangle = \overline{D}(u)|0\rangle = \left\{\frac{\sin u}{\cos u}\mathrm{e}^{2h(u)}\right\}^N |0\rangle \\
&A_{aa}(u)|0\rangle = \overline{A}_{aa}(u)|0\rangle = |0\rangle \\
&A_{21}(u)|0\rangle = \overline{A}_{21}(u)|0\rangle = 0 \\
&A_{12}(u)|0\rangle = \overline{A}_{12}(u)|0\rangle = 0 \\
&B_a(u)|0\rangle \neq 0 \qquad \overline{B}_a(u)|0\rangle \neq 0 \\
&E_a(u)|0\rangle \neq 0 \qquad \overline{E}_a(u)|0\rangle \neq 0 \\
&F(u)|0\rangle \neq 0 \qquad \overline{F}(u)|0\rangle \neq 0 \\
&C_i(u)|0\rangle = \overline{C}_i(u)|0\rangle = 0 \\
&i = 1, \ldots, 5 \qquad a, b = 1, 2.
\end{aligned}
\tag{3.6}
$$

Using the properties (3.6), and the Yang–Baxter algebra

$$\overset{2}{\mathcal{T}}{}^{-1}(-u)\mathcal{R}_{12}(u,-u)\overset{1}{\mathcal{T}}(u) = \overset{1}{\mathcal{T}}(u)\mathcal{R}_{12}(u,-u)\overset{2}{\mathcal{T}}{}^{-1}(-u) \tag{3.7}$$

and after some algebra, one can obtain

$$\tilde{B}(u)|0\rangle = W_1^-(u)B(u)\overline{B}(u)|0\rangle \tag{3.8}$$

$$\tilde{A}_{aa}(u)|0\rangle = \left\{\frac{\rho_2(u,-u)}{\rho_1(u,-u)}B(u)\overline{B}(u) + W_2^-(u)A_{aa}(u)\overline{A}_{aa}(u)\right\}|0\rangle \tag{3.9}$$

$$
\begin{aligned}
\tilde{D}(u)|0\rangle = \Bigg\{&\frac{1}{\rho_4(u,-u)}\left(K2_-(u) - \frac{\rho_2(u,-u)}{\rho_1(u,-u)}\right)\sum_{a=1}^{2} A_{aa}(u)\overline{A}_{aa}(u) \\
&+\frac{\rho_3(u,-u)}{\rho_1(u,-u)}B(u)\overline{B}(u) + W_4^-(u)D(u)\overline{D}(u)\Bigg\}|0\rangle
\end{aligned}
\tag{3.10}
$$

$$\tilde{B}_a(u)|0\rangle \neq 0 \qquad \tilde{E}_a(u)|0\rangle \neq 0 \tag{3.11}$$

$$\tilde{A}_{ab}(u)|0\rangle = 0 \qquad \tilde{F}(u) \neq 0 \tag{3.12}$$

$$\tilde{C}_i(u)|0\rangle = 0 \qquad i = 1, \ldots, 5 \qquad a \neq b = 1, 2 \tag{3.13}$$

where

$$W_1^-(u) = 1 \tag{3.14}$$

$$W_2^-(u) = -\frac{(\mathrm{e}^{-2h(u)} + \mathrm{e}^{2h(u)})\sin u\cos u(\xi_-\mathrm{e}^{h(u)}\cos u - \mathrm{e}^{-h(u)}\sin u)}{(\mathrm{e}^{2h(u)}\cos^2 u - \mathrm{e}^{-2h(u)}\sin^2 u)(\xi_-\mathrm{e}^{-h(u)}\sin u - \mathrm{e}^{h(u)}\cos u)} \tag{3.15}$$

$$
\begin{aligned}
W_4^-(u) = &\frac{(\mathrm{e}^{-2h(u)} + \mathrm{e}^{2h(u)})\sin u\cos u\sin 2u}{\cos 2u(\mathrm{e}^{-2h(u)}\cos^2 u - \mathrm{e}^{2h(u)}\sin^2 u)} \\
&\times\frac{(\mathrm{e}^{-h(u)}\xi_-\cos u - \mathrm{e}^{h(u)}\sin u)(\mathrm{e}^{h(u)}\xi_-\cos u + \mathrm{e}^{-h(u)}\sin u)}{(\mathrm{e}^{-h(u)}\cos u - \mathrm{e}^{h(u)}\xi_-\sin u)(\mathrm{e}^{h(u)}\cos u - \mathrm{e}^{-h(u)}\xi_-\sin u)}.
\end{aligned}
\tag{3.16}
$$

In this paper, for the sake of simplicity of calculation, we take the Boltzmann weight $\rho_2 = 1$. We also notice that the operators $\tilde{B}_a(u)$, $\tilde{E}_a(u)$ and $\tilde{F}(u)$ still play the roles of the creation fields, otherwise $\tilde{C}_i(u)$ are the annihilation fields. Taking into account the following transformations:

$$\tilde{A}'_{aa}(u) = \tilde{A}_{aa}(u) - \frac{\rho_2(u,-u)}{\rho_1(u,-u)}\tilde{B}(u) \tag{3.17}$$

$$\tilde{D}'(u) = \tilde{D}(u) - \frac{\rho_3(u,-u)}{\rho_1(u,-u)}\tilde{B}(u) - \frac{1}{\rho_4(u,-u)}\sum_{a=1}^{2}\tilde{A}'_{aa}(u) \tag{3.18}$$

we may express the transfer matrix (2.14) in the following way:

$$\tau(u) = \mathrm{Str}_0 K_+(u)\mathcal{T}_-(u)$$
$$= W_1^+(u)\tilde{B}(u) + W_2^+(u)\sum_{a=1}^{2}\tilde{A}'_{aa}(u) + W_4^+(u)\tilde{D}'(u) \tag{3.19}$$

where

$$W_1^+(u) = \frac{(\mathrm{e}^{-2h(u)} + \mathrm{e}^{2h(u)})\sin u\cos u\sin 2u}{\cos 2u(\mathrm{e}^{2h(u)}\cos^2 u - \mathrm{e}^{-2h(u)}\sin^2 u)} f(u)$$
$$\times\frac{(\mathrm{e}^{-h(u)}\xi_+\sin u - \mathrm{e}^{h(u)}\cos u)(\mathrm{e}^{h(u)}\xi_+\sin u - \mathrm{e}^{-h(u)}\cos u)}{(\mathrm{e}^{-h(u)}\xi_+\cos u + \mathrm{e}^{h(u)}\sin u)(\mathrm{e}^{h(u)}\xi_+\cos u + \mathrm{e}^{-h(u)}\sin u)} \tag{3.20}$$

$$W_2^+(u) = \frac{(\mathrm{e}^{-2h(u)} + \mathrm{e}^{2h(u)})\sin u\cos u}{(\mathrm{e}^{-2h(u)}\cos^2 u - \mathrm{e}^{2h(u)}\sin^2 u)} f(u)$$
$$\times\frac{(\mathrm{e}^{h(u)}\xi_+\cos u - \mathrm{e}^{-h(u)}\sin u)(\mathrm{e}^{h(u)}\xi_+\sin u - \mathrm{e}^{-h(u)}\cos u)}{(\mathrm{e}^{-h(u)}\xi_+\cos u + \mathrm{e}^{h(u)}\sin u)(\mathrm{e}^{h(u)}\xi_+\cos u + \mathrm{e}^{-h(u)}\sin u)} \tag{3.21}$$

$$W_4^+ = \frac{(\mathrm{e}^{-h(u)}\xi_+\cos u - \mathrm{e}^{h(u)}\sin u)(\mathrm{e}^{h(u)}\xi_+\cos u - \mathrm{e}^{-h(u)}\sin u)}{(\mathrm{e}^{-h(u)}\xi_+\cos u + \mathrm{e}^{h(u)}\sin u)(\mathrm{e}^{h(u)}\xi_+\cos u + \mathrm{e}^{-h(u)}\sin u)} f(u) \tag{3.22}$$

with

$$f(u) = \mathrm{e}^{-2Nh(u)}\cos^{2N} u\sin^{2N} u K1_-(u)K1_+(u). \tag{3.23}$$

Now we proceed with the key step to build up the necessary commutation relations between the diagonal and creation fields, respectively. From the RE (2.9) and definition (3.4), after many substitution steps, we can get the following important commutation relations:

$$\tilde{B}(u)\tilde{B}_a(v) = \frac{\rho_1(v,u)\rho_{10}(u,-v)}{\rho_1(v,-u)\rho_{10}(-u,-v)}\tilde{B}_a(v)\tilde{B}(u) + \mathrm{u.t.} \tag{3.24}$$

$$\tilde{D}'(u)\tilde{B}_a(v) = -\frac{\rho_7(u,-v)\rho_9(-v,-u)}{\rho_9(u,-v)\rho_8(u,v)}\tilde{B}_a(v)\tilde{D}'(u) + \mathrm{u.t.} \tag{3.25}$$

$$\tilde{A}'_{ab}(u)\tilde{B}_a(v) = -\frac{\rho_4(-v,-u)\rho_{10}(u,-v)}{\rho_1(u,-v)\rho_9(u,v)} r^{ea}_{gh}(u,-v)$$
$$\times\bar{r}^{ih}_{cb}(-v,-u)\tilde{B}_e(v)\tilde{A}'_{gi}(u) + \mathrm{u.t.} \tag{3.26}$$

$$\tilde{B}_a(u)\otimes\tilde{B}_b(v) = \frac{\rho_{10}(u,-v)\rho_4(-v,-u)}{\rho_1(u,v)\rho_{10}(v,-u)}\left\{\tilde{B}_c(v)\otimes\tilde{B}_d(u)\right.$$
$$\left.-\frac{\rho_6(u,-v)}{\rho_{10}(u,-v)}\tilde{F}(v)\vec{\xi}(I\otimes\tilde{A}(u))\right\}\cdot\bar{r}(-v,-u)$$
$$+\frac{\rho_6(v,-u)}{\rho_{10}(v,-u)}\tilde{F}(u)\vec{\xi}(I\otimes\tilde{A}(v))$$
$$+\frac{\rho_8(v,-u)\rho_6(-v,-u)}{\rho_{10}(v,-u)\rho_8(-v,-u)}[\tilde{F}(v)\tilde{B}(u) - \tilde{F}(u)\tilde{B}(v)]\cdot\vec{\xi} \tag{3.27}$$

where

$$\vec{\xi} = (0, 1, -1, 0)$$
$$\tilde{A}(u) = \begin{pmatrix}\tilde{A}_{11}(u) & \tilde{A}_{12}(u)\\ \tilde{A}_{21}(u) & \tilde{A}_{22}(u)\end{pmatrix}$$
$$r(u,-v) = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & a(u,-v) & b(u,-v) & 0\\ 0 & b(u,-v) & a(u,-v) & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{3.28}$$

$$\bar{r}(-v,-u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{a}(-v,-u) & \bar{b}(-v,-u) & 0 \\ 0 & \bar{b}(-v,-u) & \bar{a}(-v,-u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.29}$$

with the weights

$$a(u,-v) = \frac{\rho_3(u,-v)\rho_4(u,-v) - 1}{\rho_9(u,-v)\rho_{10}(u,-v)} \tag{3.30}$$

$$b(u,-v) = 1 - a(u,-v) \tag{3.31}$$

$$\bar{a}(-v,-u) = \frac{\rho_5(-v,-u)\rho_8(-v,-u) + \rho_6^2(-v,-u)}{\rho_4(-v,-u)\rho_8(-v,-u)} \tag{3.32}$$

$$\bar{b}(-v,-u) = 1 - \bar{a}(-v,-u). \tag{3.33}$$

In the commutation relations (3.24)–(3.26), we had to omit all unwanted terms because they take a large amount of space to display. It turns out that the auxiliary matrices $r(u,-v)$ and $\bar{r}(-v,-u)$ are nothing but the rational R-matrices of an isotropic six-vertex model. The structure of the auxiliary matrix is very important to solve the Hubbard-like [9,23] models with open BC that exhibit a similar structure to the auxiliary matrices equations (3.28) and (3.29). If performing the parametrization introduced in [7,24]†,

$$\tilde{x} = \frac{\sin x}{\cos x} e^{-2h(x)} - \frac{\cos x}{\sin x} e^{2h(x)} \qquad x = u, v \tag{3.34}$$

one may find that

$$a(u,-v) = \frac{U}{\tilde{u} + \tilde{v} + U} \tag{3.35}$$

$$b(u,-v) = \frac{\tilde{u} + \tilde{v}}{\tilde{u} + \tilde{v} + U} \tag{3.36}$$

$$\bar{a}(-v,-u) = \frac{U}{\tilde{u} - \tilde{v} + U} \tag{3.37}$$

$$\bar{b}(-v,-u) = \frac{\tilde{u} - \tilde{v}}{\tilde{u} - \tilde{v} + U}. \tag{3.38}$$

In view of the commutation relation (3.27), the creation operators $\tilde{B}_a$, $\tilde{E}_a$ do not interwine. So it is reasonable that the eigenvectors of the transfer matrices are generated only by the creation operators $B_a(u)$ and $F(u)$ or $E_a(u)$ and $F(u)$. Unfortunately, it seems to be very difficult to construct the explicit form of the multi-particle vector, even in the case of the Hubbard periodic chain [7]. But it does have a similar recursive relation as that for the Hubbard periodic chain. Here we prefer the n-particle vector in a formal form, namely

$$|\Phi_n(v_1,\ldots,v_n)\rangle = \Phi_n(v_1,\ldots,v_n)F^{a_1,\ldots,a_n}|0\rangle \tag{3.39}$$

where the n-particle vector $\Phi_n(v_1,\ldots,v_n)$ may be given by a recursive relation

$$\begin{aligned}\Phi_n(v_1,\ldots,v_n) = {} & \tilde{B}_a(v_1) \otimes \Phi_{n-1}(v_2,\ldots,v_n) \\ & + \sum_{j=2}^{n} [\vec{\xi} \otimes \tilde{F}(v_1)] \Phi_{n-2}(v_2,\ldots,v_{j-1},v_{j+1},\ldots,v_n) \\ & \times \tilde{B}(v_j) g_{j-1}^{(n)}(v_1,\ldots,v_n) \\ & - \sum_{j=2}^{n} [\vec{\xi} \otimes \tilde{F}(v_1)] \Phi_{n-2}(v_2,\ldots,v_{j-1},v_{j+1},\ldots,v_n) \\ & \times (I \otimes \tilde{A}(v_j)) h_{j-1}^{(n)}(v_1,\ldots,v_n).\end{aligned} \tag{3.40}$$

† The author would like to thank the referee for drawing his attention to [24].

From the commutation relation (3.27), we can conclude that $\Phi_n(v_1,\ldots,v_n)$ also satisfies the symmetry relation

$$\Phi_n(v_1,\ldots,v_j,v_{j+1},\ldots,v_n)=\frac{\rho_{10}(v_j,-v_{j+1})\rho_4(-v_{j+1},-v_j)}{\rho_1(v_j,v_{j+1})\rho_{10}(v_{j+1},-v_j)}\times\Phi_n(v_1,\ldots,v_{j+1},v_j,\ldots,v_n)\cdot\bar{r}(-v_{j+1},-v_j) \tag{3.41}$$

based on the following relation:

$$\frac{\rho_4(-v_{j+1},-v_j)}{\rho_1(v_j,v_{j+1})}\vec{\xi}\cdot\bar{r}(-v_{j+1},-v_j)=\frac{\rho_8(v_{j+1},-v_j)\rho_6(-v_{j+1},-v_j)\rho_8(-v_j,-v_{j+1})}{\rho_8(-v_{j+1},v_j)\rho_8(v_j,-v_{j+1})\rho_6(-v_j,-v_{j+1})}\cdot\vec{\xi}. \tag{3.42}$$

This symmetry, giving a restriction to the functions $h^{(n)}_{j-1}(v_1,\ldots,v_n)$ and $g^{(n)}_{j-1}(v_1,\ldots,v_n)$, is very useful in deducing the coefficients and in simplifying the unwanted terms in the eigenvalue of the transfer matrix. In fact, after performing three-particle scattering, the explicit form of these coefficients can be fixed. But checking three-particle scattering is indeed a extremely tough problem. We had to leave the coefficients to be determined later. Explicitly, we display the two-particle state:

$$\Phi_2(v_1,v_2)=\tilde{B}_{a_1}(v_1)\otimes\tilde{B}_{a_2}(v_2)-\frac{\rho_6(v_2,-v_1)}{\rho_{10}(v_2,-v_1)}\tilde{F}(v_1)\vec{\xi}(I\otimes\tilde{A}(v_2))+\frac{\rho_8(v_2,-v_1)\rho_6(-v_2,-v_1)}{\rho_{10}(v_2,-v_1)\rho_8(-v_2,-v_1)}\tilde{F}(v_1)\tilde{B}(v_2)\cdot\vec{\xi}. \tag{3.43}$$

In the above expressions, $F^{a_1,\ldots,a_n}$ are the coefficients of an arbitrary linear combination of the vectors reflecting the 'spin' degrees of freedom with $a_i=1,2$. $\vec{\xi}$ takes on the role of forbidding two spin up or two spin down at the same site. $\tilde{F}(u)$ creates a local hole pair with opposite spins. With the diagonal fields acting on the vector (3.39), we may get phenomenologically

$$\tilde{B}(u)|\Phi_n(v_1,\ldots,v_n)\rangle=\tilde{B}(u)\prod_{i=1}^{n}\frac{\rho_1(v_i,u)\rho_{10}(u,-v_i)}{\rho_1(v_i,-u)\rho_{10}(-u,-v_i)}|\Phi_n(v_1,\ldots,v_n)\rangle+\text{u.t.} \tag{3.44}$$

$$\tilde{D}'(u)|\Phi_n(v_1,\ldots,v_n)\rangle=\tilde{D}'(u)\prod_{i=1}^{n}-\frac{\rho_7(u,-v_i)\rho_9(-v_i,-u)}{\rho_9(u,-v_i)\rho_8(u,v_i)}|\Phi_n(v_1,\ldots,v_n)\rangle+\text{u.t.} \tag{3.45}$$

$$\begin{aligned}\tilde{A}'_{aa}(u)|\Phi_n(v_1,\ldots,v_n)\rangle=\tilde{A}'_{aa}(u)&\prod_{i=1}^{n}-\frac{\rho_4(-v_i,-u)\rho_{10}(u,-v_i)}{\rho_1(u,-v_i)\rho_9(u,v_i)}r_{12}(\tilde{u}+\tilde{v}_1)^{e_1a}_{h_1g_1}\\&\times r_{12}(\tilde{u}-\tilde{v}_1)^{i_1h_1}_{al_1}r_{12}(\tilde{u}+\tilde{v}_2)^{e_2g_1}_{h_2g_2}r_{12}(\tilde{u}-\tilde{v}_2)^{i_2h_2}_{i_1l_2}\ldots\\&\times r_{12}(\tilde{u}+\tilde{v}_n)^{e_ng_{n-1}}_{h_ng_n}r_{12}(\tilde{u}-\tilde{v}_n)^{i_nh_n}_{i_{n-1}l_n}|\Phi_n(v_1,\ldots,v_n)\rangle+\text{u.t.}\end{aligned} \tag{3.46}$$

It follows that

$$\begin{aligned}\tau(u)|\Phi_n(v_1,\ldots,v_n)\rangle=\Bigg\{&W_1^+(u)\tilde{B}(u)\prod_{i=1}^{n}\frac{\rho_1(v_i,u)\rho_{10}(u,-v_i)}{\rho_1(v_i,-u)\rho_{10}(-u,-v_i)}\\&+W_4^+(u)\tilde{D}'(u)\prod_{i=1}^{n}-\frac{\rho_7(u,-v_i)\rho_9(-v_i,-u)}{\rho_9(u,-v_i)\rho_8(u,v_i)}\\&+W_2^+(u)\tilde{A}'_{aa}(u)\prod_{i=1}^{n}-\frac{\rho_4(-v_i,-u)\rho_{10}(u,-v_i)}{\rho_1(u,-v_i)\rho_9(u,v_i)}\Lambda^{(1)}(\tilde{u},\{\tilde{v}_i\})\Bigg\}\\&\times|\Phi_n(v_1,\ldots,v_n)\rangle\end{aligned} \tag{3.47}$$

provided that

$$\left.\frac{W_1^+(u)\tilde{B}(u)}{W_2^+(u)\tilde{A}'_{11}(u)}\right|_{u=v_i} = -\Lambda^{(1)}(\tilde{u}=\tilde{v}_i, \{\tilde{v}_i\}) \qquad i=1,\ldots,n. \tag{3.48}$$

Here $r_{12}(u) = P \cdot r(u)$ and $\Lambda^{(1)}(\tilde{u}, \{\tilde{v}_i\})$ is the eigenvalue of the nested transfer matrix (3.50), i.e.,

$$\tau^{(1)}(\tilde{u}, \{\tilde{v}_i\}) F^{e_1,\ldots,e_n} = \Lambda^{(1)}(\tilde{u}, \{\tilde{v}_i\}) F^{e_1,\ldots,e_n} \tag{3.49}$$

where

$$\tau^{(1)}(\tilde{u}, \{\tilde{v}_i\}) = \mathrm{Tr}_0\, T^{(1)}(\tilde{u}) T^{(1)^{-1}}(-\tilde{u}). \tag{3.50}$$

The nested monodromy matrices $T^{(1)}(\tilde{u})$ and $T^{(1)^{-1}}(-\tilde{u})$ are

$$T^{(1)}(\tilde{u}) = r_{12}(\tilde{u}+\tilde{v}_1)^{e_1 a}_{h_1 g_1}, \ldots, r_{12}(\tilde{u}+\tilde{v}_n)^{e_n g_{n-1}}_{h_n g_n} \tag{3.51}$$

$$T^{(1)^{-1}}(-\tilde{u}) = r_{12}(\tilde{u}-\tilde{v}_n)^{i_n h_n}_{i_{n-1} l_n}, \ldots, r_{12}(\tilde{u}-\tilde{v}_1)^{i_1 h_1}_{a l_1}. \tag{3.52}$$

We would like to emphasize that $\tilde{B}(u)$, $\tilde{D}'(u)$ and $\tilde{A}'_{11}(u)$ are the eigenvalues of the corresponding diagonal operators acting on the pseudovacuum state, which were given in equations (3.17)–(3.23). Hereafter much care has to be paid to the differences between the variables $\tilde{u}$, $\tilde{v}$ and variables u, v, which we have to adopt on both sides of equation (3.48). So far, the eigenvalue problem of the 1D Hubbard model with boundaries reduces to solving the nested auxiliary transfer matrix (3.49) which associates with an inhomogeneous isotropic six-vertex model with open BC.

4. The nested Bethe ansatz

In this section, we proceed with the diagonalization of the auxiliary transfer matrix (3.50). Following Sklyanin's formalism [13], performing the nested Bethe ansatz has not been a difficult problem so far. It is easy to check that the $r_{12}(u)$-matrix satisfies the Yang–Baxter algebra

$$r_{12}(\tilde{u}_1-\tilde{u}_2) \stackrel{1}{T^{(1)}} (\tilde{u}_1, \{\tilde{v}_i\}) \stackrel{2}{T^{(1)}} (\tilde{u}_2, \{\tilde{v}_i\}) = \stackrel{2}{T^{(1)}} (\tilde{u}_2, \{\tilde{v}_i\}) \stackrel{1}{T^{(1)}} (\tilde{u}_1, \{\tilde{v}_i\}) r_{12}(\tilde{u}_1-\tilde{u}_2) \tag{4.1}$$

and the reflection equations

$$\begin{aligned} r_{12}(\tilde{u}_1-\tilde{u}_2) \stackrel{1}{K_-^{(1)}} (\tilde{u}_1) r_{12}(\tilde{u}_1+\tilde{u}_2) \stackrel{2}{K_-^{(1)}} (\tilde{u}_2) \\ = \stackrel{2}{K_-^{(1)}} (\tilde{u}_2) r_{12}(\tilde{u}_1+\tilde{u}_2) \stackrel{1}{K_-^{(1)}} (\tilde{u}_1) r_{12}(\tilde{u}_1-\tilde{u}_2) \end{aligned} \tag{4.2}$$

$$\begin{aligned} r_{12}(\tilde{u}_2-\tilde{u}_1) \stackrel{1}{K_+^{(1)}} (\tilde{u}_1) r_{12}(-\tilde{u}_1-\tilde{u}_2-2U) \stackrel{2}{K_+^{(1)}} (\tilde{u}_2) \\ = \stackrel{2}{K_+^{(1)}} (\tilde{u}_2) r_{12}(-\tilde{u}_1-\tilde{u}_2-2U) \stackrel{1}{K_+^{(1)}} (\tilde{u}_1) r_{12}(\tilde{u}_2-\tilde{u}_1). \end{aligned} \tag{4.3}$$

In our case, the $K_\pm^{(1)}(u) = I$. Let us define the nested monodromy matrix

$$\tilde{T}_-^{(1)}(\tilde{u}) = T^{(1)}(\tilde{u}) T^{(1)^{-1}}(-\tilde{u}) = \begin{pmatrix} \tilde{A}^{(1)}(\tilde{u}) & \tilde{B}^{(1)}(\tilde{u}) \\ \tilde{C}^{(1)}(\tilde{u}) & \tilde{D}^{(1)}(\tilde{u}) \end{pmatrix} \tag{4.4}$$

which also satisfies the RE (4.2). Using the main ingredients (4.1)–(4.4) describing the open BC compatible with the integrability of the model, and following all steps for solving the *XXZ* open chain in [13], one can present the following results:

$$\Lambda^{(1)}(\tilde{u},\{\tilde{u}_1,\ldots,\tilde{u}_M\}\{\tilde{v}_i\})|\Phi^{(1)}(\tilde{u}_l,\{\tilde{v}_i\})\rangle=\left\{\frac{2(\tilde{u}+U)}{2\tilde{u}+U}\prod_{l=1}^{M}\frac{(\tilde{u}+\tilde{u}_l)(\tilde{u}-\tilde{u}_l-U)}{(\tilde{u}-\tilde{u}_l)(\tilde{u}+\tilde{u}_l+U)}\right.$$
$$+\frac{2\tilde{u}}{2\tilde{u}+U}\prod_{i=1}^{n}b(\tilde{u}+\tilde{v}_i)b(\tilde{u}-\tilde{v}_i)$$
$$\left.\times\prod_{l=1}^{M}\frac{(\tilde{u}+\tilde{u}_l+2U)(\tilde{u}-\tilde{u}_l+U)}{(\tilde{u}-\tilde{u}_l)(\tilde{u}+\tilde{u}_l+U)}\right\}|\Phi^{(1)}(\tilde{u}_l,\{\tilde{v}_i\})\rangle \tag{4.5}$$

provided that

$$\prod_{i=1}^{n}\frac{(\tilde{u}_j+\tilde{v}_i+U)(\tilde{u}_j-\tilde{v}_i+U)}{(\tilde{u}_j+\tilde{v}_i)(\tilde{u}_j-\tilde{v}_i)}=\prod_{\substack{l=1,\\ l\neq j}}^{M}\frac{(\tilde{u}_j+\tilde{u}_l+2U)(\tilde{u}_j-\tilde{u}_l+U)}{(\tilde{u}_j+\tilde{u}_l)(\tilde{u}_j-\tilde{u}_l-U)}\qquad j=1,\ldots,M \tag{4.6}$$

which indeed ensures the cancellation of all unwanted terms in (4.5). Here the 'spin' part of the multi-particle states is given by

$$|\Phi^{(1)}(\tilde{u}_l,\{\tilde{v}_i\})\rangle=\tilde{B}^{(1)}(\tilde{u}_1),\ldots,\tilde{B}^{(1)}(\tilde{u}_M)|0\rangle^{(1)} \tag{4.7}$$

where M is the number of holes with spin down and n is the total number of holes.

Finally, if we adopt the variables $z_{\pm}(v_i)$ used in [7], i.e.,

$$z_-(v_i)=\frac{\cos v_i}{\sin v_i}\mathrm{e}^{2h(v_i)}\qquad z_+(v_i)=\frac{\sin v_i}{\cos v_i}\mathrm{e}^{2h(v_i)} \tag{4.8}$$

and make a shift on the spin rapidity $\tilde{u}_j=\tilde{\lambda}_j-U/2$, the eigenvalue of the transfer matrix (2.14) is given as

$$\tau(u)|\Phi_n(v_1,\ldots,v_n)\rangle=\left\{W_1^+(u)W_1^-(u)[z_-(u)]^{2N}\right.$$
$$\times\prod_{i=1}^{n}\frac{\sin^2 u(1+z_-(v_i)/z_+(u))(1+1/z_-(v_i)z_+(u))}{\cos^2 u(1-z_-(v_i)/z_-(u))(1-1/z_-(v_i)z_-(u))}$$
$$+W_4^+(u)W_4^-(u)[z_+(u)]^{2N}\prod_{i=1}^{n}\frac{\sin^2 u(1+z_-(v_i)z_-(u))(1+z_-(u)/z_-(v_i))}{\cos^2 u(1-z_-(v_i)z_+(u))(1-z_+(u)/z_-(v_i))}$$
$$+W_2^+(u)W_2^-(u)\frac{2(\tilde{u}+U)}{2\tilde{u}+U}$$
$$\times\prod_{i=1}^{n}\frac{\sin^2 u(1+z_-(v_i)/z_+(u))(1+1/z_-(v_i)z_+(u))}{\cos^2 u(1-z_-(v_i)/z_-(u))(1-1/z_-(v_i)z_-(u))}$$
$$\times\prod_{l=1}^{M}\frac{(\tilde{u}+\tilde{\lambda}_l-U/2)(\tilde{u}-\tilde{\lambda}_l-U/2)}{(\tilde{u}-\tilde{\lambda}_l+U/2)(\tilde{u}+\tilde{\lambda}_l+U/2)}$$
$$+W_2^+(u)W_2^-(u)\frac{2\tilde{u}}{2\tilde{u}+U}\prod_{i=1}^{n}\frac{\sin^2 u(1+z_-(v_i)z_-(u))(1+z_-(u)/z_-(v_i))}{\cos^2 u(1-z_-(v_i)z_+(u))(1-z_+(u)/z_-(v_i))}$$
$$\left.\times\prod_{l=1}^{M}\frac{(\tilde{u}+\tilde{\lambda}_l+3U/2)(\tilde{u}-\tilde{\lambda}_l+3U/2)}{(\tilde{u}-\tilde{\lambda}_l+U/2)(\tilde{u}+\tilde{\lambda}_l+U/2)}\right\}|\Phi_n(v_1,\ldots,v_n)\rangle \tag{4.9}$$

provided that

$$\zeta(v_i, \xi_+)\zeta(v_i, \xi_-)[z_-(v_i)]^{2N} = \prod_{l=1}^{M} \frac{(\tilde{v}_i + \tilde{\lambda}_l - U/2)(\tilde{v}_i - \tilde{\lambda}_l - U/2)}{(\tilde{v}_i - \tilde{\lambda}_l + U/2)(\tilde{v}_i + \tilde{\lambda}_l + U/2)} \tag{4.10}$$

$$\prod_{i=1}^{n} \frac{(\tilde{\lambda}_j + \tilde{v}_i + U/2)(\tilde{\lambda}_j - \tilde{v}_i + U/2)}{(\tilde{\lambda}_j + \tilde{v}_i - U/2)(\tilde{\lambda}_j - \tilde{v}_i - U/2)} = \prod_{\substack{l=1,\\ l\neq j}}^{M} \frac{(\tilde{\lambda}_j + \tilde{\lambda}_l + U)(\tilde{\lambda}_j - \tilde{\lambda}_l + U)}{(\tilde{\lambda}_j + \tilde{\lambda}_l - U)(\tilde{\lambda}_j - \tilde{\lambda}_l - U)}$$

$$j = 1, \ldots M \qquad i = 1, \ldots, n \tag{4.11}$$

where

$$\zeta(u, \xi_\pm) = \frac{e^{-h(u)}\xi_\pm \sin u - e^{h(u)} \cos u}{e^{h(u)}\xi_\pm \cos u - e^{-h(u)} \sin u}. \tag{4.12}$$

If we express the variable $z_-(u_i)$ in terms of the momenta k_i (hole) by $z_-(u_i) = e^{ik_i}$, from the relation (2.15), the energy is given by

$$E_n = \xi_- + \xi_+ - (N/2 - n)U - \sum_{i=1}^{n} 4\cos k_i. \tag{4.13}$$

Equations (4.9)–(4.13) constitute our main results of this paper. Now let us adopt the conventional notations, using the momenta k_i instead of the charge rapidity $\tilde{v}_i$ via the relation (3.34) and making a scaling on spin rapidity $\tilde{\lambda}_j$ as $\lambda_j = -\frac{i}{2}\tilde{\lambda}_j$. Then the Bethe equations (4.10) and (4.11) are

$$\zeta(k_i, \xi_+)\zeta(k_i, \xi_-)e^{i2Nk_i} = \prod_{l=1}^{M} \frac{(\sin k_i - \lambda_l - \frac{iU}{4})(\sin k_i + \lambda_l - \frac{iU}{4})}{(\sin k_i - \lambda_l + \frac{iU}{4})(\sin k_i + \lambda_l + \frac{iU}{4})} \tag{4.14}$$

$$\prod_{i=1}^{n} \frac{(\sin k_i - \lambda_j - \frac{iU}{4})(\sin k_i + \lambda_j - \frac{iU}{4})}{(\sin k_i - \lambda_j + \frac{iU}{4})(\sin k_i + \lambda_j + \frac{iU}{4})} = \prod_{\substack{l=1,\\ l\neq j}}^{M} \frac{(\lambda_j - \lambda_l + \frac{iU}{2})(\lambda_j + \lambda_l + \frac{iU}{2})}{(\lambda_j - \lambda_l - \frac{iU}{2})(\lambda_j + \lambda_l - \frac{iU}{2})}$$

$$j = 1, \ldots M \qquad i = 1, \ldots, n \tag{4.15}$$

with

$$\zeta(k_i, \xi_\pm) = \frac{\xi_\pm - e^{ik_i}}{\xi_\pm e^{ik_i} - 1}. \tag{4.16}$$

So far our results can be incorporated into the notation used in [19], which provides us with a detailed computation of the low-lying spectrum for the 1D Hubbard model with boundary fields based on the coordinate Bethe ansatz solution. Their discussions are also valid, apart from the different expression for the boundary K-matrices. It is found that the boundary fields are indeed nontrivial to the ground-state properties as well as the low-lying spectrum. The function $\zeta(k_i, \xi_\pm)$ contributes as a phase shift to the density of the roots of the rapidities. The boundary fields $\xi_\pm$, acting as the impurity parameters, change the band filling, the boundary surface energy and the mesoscopic effects as well.

5. Conclusion

We have formulated the algebraic Bethe ansatz solution for the 1D Hubbard model with open boundaries. The Bethe ansatz equations, the eigenvalue of the transfer matrix and the energy spectrum have also been given. Comparing our results with the coordinate Bethe ansatz solution [19], the Bethe ansatz equations (4.10) and (4.11) coincide with those obtained in [19]. In addition to this, we presented explicitly the eigenvalue of the transfer matrix and the main structure of the n-particle eigenvectors. This paper seems to bridge a gap in solving the Hubbard-like open chains by the quantum R-matrix approach with or without the additive property of spectral parameters, such as the 1D Bariev open chain [25], $U_q[\mathrm{Osp}(2|2)]$ electronic system [23, 26], etc. The results obtained provided us with a starting point to study the thermodynamical properties and correlation functions for the model [6, 20]. We notice that the Bethe equations (4.14) and (4.15) would be reduced to the purely doubling ones for the 1D Hubbard model with periodic BC if the boundary parameters $\xi_\pm \to \infty$. In such a case, the model exhibits the closed BC, which preserves the quantum group invariance [27, 28]. We also notice that, if we add the chemical potential term $\nu \sum_{j=1}^{N} \sum_s (n_{js} - \frac{1}{2})$ to the Hamiltonian (2.1), the integrability of the model requires that the associated quantum R-matrix with an extra free parameter [29], which does not have crossing unitarity, should satisfy the new RE. But the new class of boundary $K_\pm$-matrices [30] leads to the same Bethe ansatz equations as (4.10) and (4.11), in spite of the different engenvalues of the transfer matrix. The multiparametric quantum R-matrix [31] does provide the Bethe equations with an additional phase shift for the quantum integrable periodic chain. Nevertheless, as far as we know, it is still not clear if there exist nontrivial phases appearing in the Bethe equations for the quantum open chain associated with the multiparametric R-matrix. It seems to be more interesting that the 1D Hubbard model with a special kind of twisted BC can be regarded as a spin ladder model [32]. On the other hand, if we add the Kondo impurities [11, 33, 34] $J \sum_{ss'} a_s^\dagger \boldsymbol{\sigma}_{ss'} a_{s'} \cdot \boldsymbol{S}$ to each boundary, the model is also integrable. If we embed an inhomogeneous Lax operator [4] carrying both charge and spin degrees of freedom into the open chain, the Hamiltonian (2.1) will have additional impurity terms leading to an additional phase shift in the Bethe ansatz equations. The impurity parameter changes the finite-size correction spectrum in a different way from the boundary fields. The algebraic Bethe ansatz structure of this paper provides a clear picture for different impurity embedding. We hope that we shall present a class of integrable impurities carrying both charge and spin degrees of freedom for the 1D Hubbard model with boundary fields in the near future.

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Appendix

We display the quantum $\mathcal{R}(u, v)$-matrix of the 1D Hubbard model below [3, 4]:

$$\begin{pmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\mathrm{i}\rho_{10} & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\mathrm{i}\rho_{10} & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_8 & 0 & 0 & \mathrm{i}\rho_6 & 0 & 0 & -\mathrm{i}\rho_6 & 0 & 0 & \rho_3 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 & \mathrm{i}\rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\rho_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathrm{i}\rho_6 & 0 & 0 & -\rho_7 & 0 & 0 & -\rho_5 & 0 & 0 & -\mathrm{i}\rho_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 \\
0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}\rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mathrm{i}\rho_6 & 0 & 0 & -\rho_5 & 0 & 0 & -\rho_7 & 0 & 0 & \mathrm{i}\rho_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\rho_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}\rho_9 & 0 & 0 & \rho_2 & 0 \\
0 & 0 & 0 & \rho_3 & 0 & 0 & -\mathrm{i}\rho_6 & 0 & 0 & \mathrm{i}\rho_6 & 0 & 0 & \rho_8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\rho_{10} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & -\mathrm{i}\rho_{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_1
\end{pmatrix}$$

with the Boltzmann weights

$$\rho_1 = (\cos u \cos v \mathrm{e}^{l} + \sin v \sin u \mathrm{e}^{-l})\rho_2$$
$$\rho_4 = (\cos u \cos v \mathrm{e}^{-l} + \sin v \sin u \mathrm{e}^{l})\rho_2$$
$$\rho_9 = (\sin u \cos v \mathrm{e}^{-l} - \sin v \cos u \mathrm{e}^{l})\rho_2$$
$$\rho_{10} = (\sin u \cos v \mathrm{e}^{l} - \sin v \cos u \mathrm{e}^{-l})\rho_2$$
$$\rho_3 = \frac{(\cos u \cos v \mathrm{e}^{l} - \sin v \sin u \mathrm{e}^{-l})}{\cos^2 u - \sin^2 v}\rho_2$$
$$\rho_5 = \frac{(\cos u \cos v \mathrm{e}^{-l} - \sin v \sin u \mathrm{e}^{l})}{\cos^2 u - \sin^2 v}\rho_2$$
$$\rho_6 = \frac{\mathrm{e}^{-h}(\cos u \sin u \mathrm{e}^{l} - \sin v \cos v \mathrm{e}^{-l})}{\cos^2 u - \sin^2 v}\rho_2$$

and

$$\rho_8 = \rho_1 - \rho_3$$
$$\rho_7 = \rho_4 - \rho_5$$
$$l = h(u) - h(v)$$
$$h = h(u) + h(v)$$

which enjoy the following identities:

$$\rho_4\rho_1 + \rho_9\rho_{10} = 1$$
$$\rho_1\rho_5 + \rho_3\rho_4 = 2$$
$$\rho_6^2 = \rho_3\rho_5 - 1$$
$$\rho_6^2 = \rho_9\rho_{10} + \rho_7\rho_8.$$

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